Classification of the Entangled States of $2 \times N \times N$

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Abstract

We develop a novel method in classifying the multipartite entanglement state of $2 \times N \times N$ under stochastic local operation and classical communication. In this method, all inequivalent classes of true entangled state can be assorted directly without knowing the classification information of lower dimension ones for any given dimension N. It also gives a nature explanation for the non-local parameters remaining in the entanglement classes while $N \geq 4$.

1 Introduction

Entanglement is at the heart of the quantum information theory (QIT) and is now thought as a physical resource to realize quantum information tasks, such as quantum cryptography [1, 2], superdense coding [3, 4], and quantum computation [5], etc. Moreover, the study of entanglement may also improve our knowledge about quantum nonlocality [6]. Among it the investigation on the classification of multipartite entanglement is of particular interest in QIT. According to QIT, two quantum states can be employed to carry on the same task while they are thought to be equivalent in the meaning of mutually convertible under Stochastic Local Operations and Classical Communication (SLOCC) [7].

Nevertheless, in practice the classification of multipartite entanglement in high dimension in the Hilbert space is generally mathematically difficult [8]. It was found that

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the matrix decomposition method [9, 10] keeps to be a useful tool as in the two-partite case. A widely adopted philosophy in dealing with this issue is first to classify the state in lower dimension (or less partite) and then extend to the higher dimension [11, 12] (or more partite [13, 14]) cases in an inductive way. However, nontrivial aspect emerges as the dimension increases, i.e. some non-local parameters may nest in the entangled states [15, 16]. In recent years, investigations on the classification of $2 \times M \times N$ states were performed [11, 12], where M and N are dimensions of two partites in three-partite entangled states. Based on the "range criterion", an iterated method was introduced to determine all classes of true entangled states of the $2 \times M \times N$ system in Refs.[11, 12]. In this scenario the entanglement classes of high dimensional states can be obtained through the low dimensional ones. That is, first generate all the possible entanglement classes under invertible local operator (ILO) by the classification information of lower dimensional ones, then use the "range criterion" to find out the inequivalent classes of true entanglement among all the possible entangled classes, which tends to be a formidable task with the increase of dimensions. The main trait of this scenario is that the lower dimensional entanglement classes are prerequisite for the follow-up classification. As mentioned in Ref.[12] the classification of entangled state of $2 \times M \times N$ becomes more subtle when $M = N$. In this case the permutations of the two N-dimensional partites may be assorted into different classes.

In this work, we present a straightforward method in fully classifying the entanglement states in $2 \times N \times N$ configuration. The asymmetry of the two N-dimensional partites shows up in one of the classes. We develop a cubic grid form for the quantum state, in which the entangled classes that have continuous parameters can be explained naturally. This gives an instructive insight on the entanglement classes of 4 or more partites which also have non-local parameters [7, 14].

The paper is arranged as follows: after the introduction section, we represent the entangled state in a general form in section 2, by which the true entangled state of $2 \times N \times N$ can be expressed in a matrix pair. With the definitions given in section 2, the true entangled state of $2 \times N \times N$ can be fully classified according to the theorems given in section 3. In section 4, two examples on how to employ the novel classification method are presented. We show, in a typical case of $2 \times 5 \times 5$, that the non-local parameters generally may exist in high dimensional or multi-partite entangled state. The last section is remained for a brief summary.

2 Representation of the entangled states of $2 \times N \times N$

An arbitrary two-partite state in dimension of M times N can be expressed in the following form

$$\begin{aligned} |\Psi_{M\times N}\rangle &= \gamma_{11}|11\rangle + \gamma_{12}|12\rangle + \cdots \gamma_{1N}|1N\rangle + \\ & \quad \gamma_{21}|21\rangle + \gamma_{22}|22\rangle + \cdots + \gamma_{2N}|2N\rangle + \\ & \quad \vdots \\ & \quad \gamma_{M1}|M1\rangle + \gamma_{M2}|M2\rangle + \cdots \gamma_{MN}|MN\rangle \ , \end{aligned} \tag{1}$$

where $\gamma_{ij} \in \mathbb{C}$, are a series of complex numbers. Eq.(1) can be further expressed in a more compact form

$$\begin{aligned} |\Psi_{M\times N}\rangle &= (|1\rangle, |2\rangle, \cdots, |M\rangle) \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{M1} & \gamma_{M2} & \cdots & \gamma_{MN} \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \\ \vdots \\ |N\rangle \end{pmatrix} \\ &\equiv \psi_1^{\mathrm{T}} \Gamma_{\{i,j\}} \psi_2 = Tr[\Gamma_{\{i,j\}} \ \psi_2 \otimes \psi_1^{\mathrm{T}}] \ . \end{aligned} \tag{2}$$

Here, $\Gamma_{\{i,j\}}$ denotes the $M \times N$ complex matrix, which can also be treated as a tensor of rank two, and $\otimes$ is the symbol of direct product. Obviously, the feature of a $M \times N$ pure state is characterized by the rank-two tensor $\Gamma_{\{i,j\}}$. Similarly, the state of $2 \times N \times N$ may be expressed in a traced form

$$|\Psi_{2\times N\times N}\rangle = Tr[\Gamma_{\{i,j,k\}} \ \psi_2 \otimes \psi_1^{\mathrm{T}} \otimes \psi_0^{\mathrm{T}}] \ , \tag{3}$$

where, ψ_0 is a 2-dimensional vector and $\psi_{1,2}$ are N-dimensional vectors, representing the constituent states in Hilbert space. The $2 \times N \times N$ matrix $\Gamma_{\{i,j,k\}}$, which can also be taken as a rank-three tensor, reads

$$\Gamma_{\{i,j,k\}} = \begin{pmatrix} \gamma_{111} & \gamma_{112} & \cdots & \gamma_{11N} \\ \gamma_{121} & \gamma_{122} & \cdots & \gamma_{12N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1N1} & \gamma_{1N2} & \cdots & \gamma_{1NN} \\ \gamma_{211} & \gamma_{212} & \cdots & \gamma_{21N} \\ \gamma_{221} & \gamma_{222} & \cdots & \gamma_{22N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{2N1} & \gamma_{2N2} & \cdots & \gamma_{2NN} \end{pmatrix} = \begin{pmatrix} \Gamma_{\{1,l,m\}} \\ \Gamma_{\{2,l,m\}} \end{pmatrix} \equiv \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \ . \tag{4}$$

Here, $\Gamma_{\{1,l,m\}}$ and $\Gamma_{\{2,l,m\}}$ are in fact tensors of rank two, which are represented by $N \times N$ complex matrices $\Gamma_{1,2}$. The Γ_1 and Γ_2 stand for the upper and lower N-line blocks of matrix $\Gamma_{\{i,j,k\}}$, respectively.

From (3) and (4) we know that the information of the state $2 \times N \times N$ is involved in the matrix pair $(\Gamma_1\,, \Gamma_2)$. Therefore, in the aim of classification we can specify the typical entangled state by a 'matrix vector', that is

$$|\Psi_{2\times N\times N}\rangle \;\doteq\; \left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array}\right)\;, \tag{5}$$

where, the symbol $\doteq$ stands for "is represented by".

Generally speaking, two states are said to be SLOCC equivalent if they are connected by ILOs [7]. For instance, in the case of bipartite entanglement, suppose the two partites are transformed under two invertible local operators P and Q, i.e. $\psi_1' = P^{\mathrm{T}}\psi_1, \psi_2' = Q\psi_2$, then from Eq.(2) a SLOCC equivalent state to this bipartite entangled state reads as

$$\begin{aligned} |\Psi'_{M\times N}\rangle &= \psi_1'^{\mathrm{T}}\Gamma_{\{i,j\}}\psi_2' \\ &= Tr[P\Gamma_{\{i,j\}}Q\;\psi_2\otimes\psi_1^{\mathrm{T}}] \\ &= \psi_1^{\mathrm{T}}\Gamma'_{\{i,j\}}\psi_2\;. \end{aligned} \tag{6}$$

From above expression, we see that two SLOCC equivalent states are in fact connected only by the transformation of the matrix $\Gamma_{\{i,j\}}$ in Eq.(2) like

$$\Gamma'_{\{i,j\}} = P\;\Gamma_{\{i,j\}}\;Q\;. \tag{7}$$

Similarly, two SLOCC equivalent $2 \times N \times N$ states $|\Psi'_{2\times N\times N}\rangle$ and $|\Psi_{2\times N\times N}\rangle$ are also connected by the transformation of the matrix $\Gamma_{\{i,j,k\}}$ in Eq.(4), i.e.,

$$\begin{aligned} |\Psi'_{2\times N\times N}\rangle &= Tr[\Gamma_{\{i,j,k\}}\;\psi_2'\otimes\psi_1'^{\mathrm{T}}\otimes\psi_0'^{\mathrm{T}}] \\ &= Tr[T\otimes P\Gamma_{\{i,j,k\}}Q\;\psi_2\otimes\psi_1^{\mathrm{T}}\otimes\psi_0^{\mathrm{T}}] \\ &= Tr[\Gamma'_{\{i,j,k\}}\;\psi_2\otimes\psi_1^{\mathrm{T}}\otimes\psi_0^{\mathrm{T}}]\;, \end{aligned} \tag{8}$$

where

$$\Gamma'_{\{i,j,k\}} = \left(\begin{array}{c} \Gamma'_{\{1,j,k\}} \\ \Gamma'_{\{2,j,k\}} \end{array}\right) = T\left(\begin{array}{c} P\Gamma_1 Q \\ P\Gamma_2 Q \end{array}\right)\;. \tag{9}$$

Here, T is any invertible 2×2 matrix which acts on ψ_0; P and Q are two invertible $N \times N$ matrices acting on ψ_1 and ψ_2, respectively. The transformation of the first partite T reads as

$$\left(\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array}\right)\left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array}\right)\;. \tag{10}$$

For brevity, as in (5), equation (8) can be formulated as

$$|\Psi'_{2\times N\times N}\rangle = \left(\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array}\right)\left(\begin{array}{c} P\Gamma_1 Q \\ P\Gamma_2 Q \end{array}\right)\;, \tag{11}$$

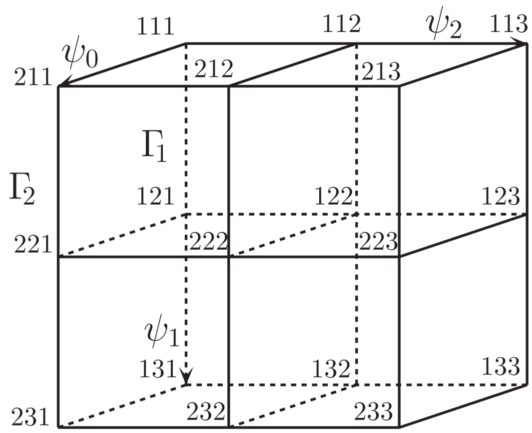

Figure 1: The pictorial description of $2 \times 3 \times 3$ state, where each node corresponds to a base vector. Assigning a coefficient to the base vector, we then obtain the corresponding matrix element of Γ_1 or Γ_2.

where ψ s are suppressed.

To give a pictorial description of the quantum state, we take $2 \times 3 \times 3$ case as an example, see Figure 1. The matrices Γ_1 and Γ_2 are placed in parallel in rear and front of the cubic, respectively. Of the cubic grid, each node corresponds to an element in the matrix pair (Γ_1, Γ_2) in Eq.(5) .

In matrix algebra, every ILO which acts on a given matrix can be decomposed as a series of products of elementary operations on the matrix, and there exist three such elementary operations [17]. Therefore, the matrices T, P and Q in Eq.(8), which connect the two equivalent wave functions, can be decomposed as such sequence of elementary operations. In the pictorial language, here the three types of elementary operation correspond to three types of manipulation of the cubic grid: **type 1**, interchange of two surfaces; **type 2**, multiplication of one surface by a nonzero scalar; **type 3**, addition of a scalar multiple of one surface to another surface. Specifically, T is responsible for the elementary operations between front and rear; P for upper and lower, Q for left and right surfaces, respectively.

According to the common definition [7], a true $2 \times N \times N$ entangled state requires the following conditions

$$r(\rho_{\psi_0}) = 2,\ r(\rho_{\psi_1}) = r(\rho_{\psi_2}) = N \ , \tag{12}$$

to be true, where $\rho_i = \mathrm{Tr}_{jk}(\rho_{ijk})$ being the reduced density matrix. Hereafter, we denote r to be the rank of matrix. In Quantum Mechanics, to each state there corresponds a unique state operator, the density matrix. In the representation of matrix pair the density

matrix(elements) can be expressed as

$$\rho_{\psi_0,\psi_1,\psi_2} = \Gamma_{ijk}\Gamma^*_{i'j'k'} \ , \tag{13}$$

where $i, i' = 1,2; j, j' = 1,2,\cdots,N; k, k' = 1,2,\cdots,N$. The corresponding reduced density matrix is (taking ψ_2 as an example)

$$\begin{aligned}\rho_{\psi_2} &= \mathrm{Tr}_{\psi_0,\psi_1}(\rho_{\psi_0,\psi_1,\psi_2}) \\ &= \sum_{ij}\Gamma_{ijk}\Gamma^*_{ijk'} \\ &= \sum_i(\Gamma_i^\dagger\Gamma_i)_{k'k} \ .\end{aligned} \tag{14}$$

If $\mathrm{Det}(\rho_{\psi_2}) \neq 0$, then we know $r(\rho_{\psi_2}) = N$.

3 Classification of the tripartite entangled state $2 \times N \times N$

With the above preparation, we can now proceed to classify the $2 \times N \times N$ state. Generically, the whole space of the state (Γ_1, Γ_2) can be partitioned into numbers of inequivalent sets by different l and n.

$$(\Gamma_1, \Gamma_2) = \{C_{n,\,l}\} \ , \tag{15}$$

$$C_{n,\,l} = \{(\Gamma_1,\Gamma_2)|\ r_{\max}(\alpha_1\Gamma_1 + \beta_1\Gamma_2) = n, r_{\min}(\alpha_2\Gamma_1 + \beta_2\Gamma_2) = l\} \ , \tag{16}$$

where $\alpha_i, \beta_i \in \mathbb{C}$ and $|\alpha_i| + |\beta_i| \neq 0$; $l \in [0,n]$ and $n \in [0,N]$; $r_{\max}$ and $r_{\min}$ are maximum and minimum ranks of matrices for all possible values of α_i and β_i. From the definition of (16), there is no common element in different sets, i.e. $C_{n,\,l} \cap C_{m,\,k} = C_{n,\,l}\ \delta_{m,n}\delta_{l,k}$. Obviously, every (entangled) state (Γ_1, Γ_2) must lie in one of the subspaces of the set $\{C_{n,\,l}\}$ with certain n and l, which in principle can be determined via the transformation of equation (11), since one can always classify a set by certain rules voluntarily. Here, the criteria $r_{\max}(\alpha_1\Gamma_1 + \beta_1\Gamma_2) = n$ and $r_{\min}(\alpha_2\Gamma_1 + \beta_2\Gamma_2) = l$ attribute to the group SL(2, $\mathbb{C}$) transformation T in (9). Note that in case $\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ is non-invertible, when r_{max}, r_{min}, Rank(Γ_1), and Rank(Γ_2) are all equal in magnitude, its function can be fulfilled by a unit matrix.

Suppose $(\Gamma_1,\ \Gamma_2) \in C_{n,\ l}$, $(\bar{\Gamma}_1,\ \bar{\Gamma}_2) \in C_{\bar{n},\ \bar{l}}$, and

$$\mathrm{Rank}\left[\mathrm{O_A}\begin{pmatrix}\Gamma_1 \\ \Gamma_2\end{pmatrix}\right] = \mathrm{Rank}\left[\begin{pmatrix}\alpha_1 & \beta_1 \\ \alpha_2 & \beta_2\end{pmatrix}\begin{pmatrix}\Gamma_1 \\ \Gamma_2\end{pmatrix}\right] = \mathrm{Rank}\begin{pmatrix}\Gamma_1' \\ \Gamma_2'\end{pmatrix} = \begin{pmatrix} n \\ l\end{pmatrix} \ , \tag{17}$$

$$\text{Rank}\left[\bar{\text{O}}_{\text{A}}\left(\begin{array}{c}\bar{\Gamma}_1\\ \bar{\Gamma}_2\end{array}\right)\right] = \text{Rank}\left[\left(\begin{array}{cc}\bar{\alpha}_1 & \bar{\beta}_1\\ \bar{\alpha}_2 & \bar{\beta}_2\end{array}\right)\left(\begin{array}{c}\bar{\Gamma}_1\\ \bar{\Gamma}_2\end{array}\right)\right] = \text{Rank}\left(\begin{array}{c}\bar{\Gamma}_1'\\ \bar{\Gamma}_2'\end{array}\right) = \left(\begin{array}{c}\bar{n}\\ \bar{l}\end{array}\right) , \tag{18}$$

where O_A and $\bar{\text{O}}_\text{A}$ are invertible operators; the "Rank" denotes the rank operation on Γ matrices in upper and lower blocks separately, there will have no invertible matrix O_I exist, which enables

$$O_I\left(\begin{array}{c}\bar{\Gamma}_1\\ \bar{\Gamma}_2\end{array}\right) = \left(\begin{array}{c}\Gamma_1\\ \Gamma_2\end{array}\right) \tag{19}$$

in case $n \neq \bar{n}$ or $l \neq \bar{l}$, i.e. $(\Gamma_1,\ \Gamma_2)$ and $(\bar{\Gamma}_1,\ \bar{\Gamma}_2)$ are SLOCC inequivalent. If the operator O_I exists, substituting (19) into (17) we may have

$$\text{Rank}\left[\text{O}_\text{A}\text{O}_\text{I}\left(\begin{array}{c}\bar{\Gamma}_1\\ \bar{\Gamma}_2\end{array}\right)\right] = \left(\begin{array}{c}n\\ l\end{array}\right) , \tag{20}$$

and from (18) one knows that $n \leq \bar{n}$ and $l \geq \bar{l}$. Similarly, since O_I is an invertible operator(matrix), one may also get $\bar{n} \leq n$ and $\bar{l} \geq l$, and hence, $\bar{n} = n$ and $\bar{l} = l$. From the above arguments, two states, the matrix pairs $(\Gamma_1,\ \Gamma_2)$ and $(\bar{\Gamma}_1,\ \bar{\Gamma}_2)$, connected via invertible operator belong to the same subset $C_{n,\,l}$.

Therefore, the entangled classes in set $\{C_{n,\,l}\}$ with different n and l are SLOCC inequivalent, and the question of performing a complete classification on entangled states now turns to how to classify the entangled states in subset $C_{n,\,l}$.

3.1 Classification on set $C_{n,l}$ with $n = N$

From the definition of $C_{N,l}$ we know that if $(\Gamma_1, \Gamma_2) \in C_{N,l}$, then there exist an invertible operator T which enables

$$T\left(\begin{array}{c}\Gamma_1\\ \Gamma_2\end{array}\right) = \left(\begin{array}{cc}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{array}\right)\left(\begin{array}{c}\Gamma_1\\ \Gamma_2\end{array}\right) = \left(\begin{array}{c}\Gamma_1'\\ \Gamma_2'\end{array}\right) , \tag{21}$$

where Γ_1' has the maximum rank N and Γ_2' has minimum rank l. According to matrix algebra, in principle one can find invertible operators P, Q and S which further transform the (Γ_1', Γ_2') in the following form

$$SP \otimes QS^{-1}\left(\begin{array}{c}\Gamma_1'\\ \Gamma_2'\end{array}\right) \equiv \left(\begin{array}{c}SP\Gamma_1'QS^{-1}\\ SP\Gamma_2'QS^{-1}\end{array}\right) = \left(\begin{array}{c}E\\ J\end{array}\right) . \tag{22}$$

Here, $r(J) = r_{\min}(\alpha_2\Gamma_1 + \beta_2\Gamma_2)$ with J a matrix in the Jordan canonical form. A typical form of J reads

$$J = \left(\begin{array}{cccc}J_{n_1}(\lambda_1) & 0 & \cdots & 0\\ 0 & J_{n_2}(\lambda_2) & & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & J_{n_k}(\lambda_k)\end{array}\right) , \tag{23}$$

in which $J_{n_i}(\lambda_i)$ is a $n_i \times n_i$ matrix which has the following form

$$J_{n_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & & 0 & 0 \\ 0 & 0 & \lambda_i & & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix} . \tag{24}$$

In all, for every $(\Gamma_1, \Gamma_2) \in C_{N,\, l}$, there exists an ILO transformation, like

$$\begin{pmatrix} E \\ J \end{pmatrix} = T \otimes P \otimes Q \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} . \tag{25}$$

Provided $r(J) = N$, we know that the rank of the matrix $J' \equiv (J - \lambda_i E)$, with λ_i being any eigenvalue of J, must be less than that of J's. This conclusion contradicts with the proviso of J having the minimum rank, since J' and J are correlated through an invertible operator, let's say T',

$$\begin{pmatrix} E \\ J' \end{pmatrix} = T' \begin{pmatrix} E \\ J \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda_i & 1 \end{pmatrix} \begin{pmatrix} E \\ J \end{pmatrix} . \tag{26}$$

From above arguments, one observes that the rank of J is less than N, i.e. $l \leq N - 1$. In the special case of $N = 2$, this observation agrees with the proposition given in Refs. [13, 18]. From Eq.(25) $c_{N,\, l} = (E, J)$ is equivalent to $C_{N,\, l}$ under the joint invertible transformations of T, P, and Q, that means the classification on $C_{N,\, l}$ can be simply performed on $c_{N,\, l}$.

From Eq.(14) one can find that for the quantum state (matrix pair) in $c_{N,l}$

$$\mathrm{Det}(\rho_{\psi_j}) = \prod_i [\sum_{m=0}^{n_i} \frac{(1 + |\lambda_i|^2)^m}{(n_i - m)!} \; f_{m+1}^{(n_i - m)}(x)\Big|_{x=0} \,] \neq 0 \tag{27}$$

with $f_n(x) = \left[\frac{1-x}{1-x-x^2}\right]^n$. Here, $j = \psi_1, \psi_2$ and n_i, λ_i are defined in Eq.(24). This tells that $r(\rho_{\psi_1}) = r(\rho_{\psi_2}) = N$. When $l \neq 0$ we readily have $r(\rho_{\psi_0}) = 2$. This means the state in $c_{N,l}$ is true entangled $2 \times N \times N$ state, while $l \neq 0$. Otherwise it will not be a true entangled $2 \times N \times N$ state, which is beyond our consideration.

Theorem 1 $\forall\ (E, J) \in c_{N,\, l}$, *the set* $c_{N,\, l}$ *is of the classification of* $C_{N,\, l}$ *under SLOCC:*
(i) *if two states in* $C_{N,\, l}$ *are SLOCC equivalent, then they can be transformed into the same matrix vector* (E, J)*;*
(ii) *matrix vector* (E, J) *is unique in* $c_{N,\, l}$ *up to a trivial transformation, that is if* (E, J') *is SLOCC equivalent with* (E, J)*, then* $(E, J') = (E, J + \lambda E)$ *with* λ *being an arbitrary complex number.*

Proof:
(i) Suppose there exists the transformation

$$\left(\begin{array}{c} \Gamma_1' \\ \Gamma_2' \end{array} \right) = T' \otimes P' \otimes Q' \left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array} \right) , \tag{28}$$

according to equation (25)

$$\left(\begin{array}{c} E \\ J \end{array} \right) = T \cdot T'^{-1} \otimes P \cdot P'^{-1} \otimes Q \cdot Q'^{-1} \left(\begin{array}{c} \Gamma_1' \\ \Gamma_2' \end{array} \right) . \tag{29}$$

(ii) Suppose

$$\left(\begin{array}{c} E \\ J' \end{array} \right) = T' \otimes P' \otimes Q' \left(\begin{array}{c} E \\ J \end{array} \right) , \tag{30}$$

as noted beneath the Eq.(23) we have $l \leq N-1$, and it tells that there are no zero elements in the pivot of T'. Then, T' can be decomposed as lower and upper triangular(LU) forms [19]

$$\left(\begin{array}{cc} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \cdot \left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) , \tag{31}$$

where $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ and both matrices on the righthand side are nonsingular. Now Eq.(30) becomes

$$\left(\begin{array}{c} E \\ J' \end{array} \right) = O_1 O_2 \left(\begin{array}{c} E \\ J \end{array} \right) \tag{32}$$

with

$$O_1 = \left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) , \;\; O_2 = \left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) \otimes P' \otimes Q' . \tag{33}$$

Here, according to the definition of operator Q in Eqs.(11) and (22), Q' is applied to matrix vector from the right hand side. Thus the operator O_2 acts on (E, J) in following way

$$O_2 \left(\begin{array}{c} E \\ J \end{array} \right) \;\; = \;\; P' \left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) \left(\begin{array}{c} E \\ J \end{array} \right) Q' , \tag{34}$$

where $J = J(\lambda_i) = \oplus_i J_{n_i}(\lambda_i)$. The Eq.(32) now gives two independent equations

$$E \;\; = \;\; P'(\alpha E + \beta J) Q' , \tag{35}$$
$$J' \;\; = \;\; \lambda E + \gamma P' J Q' . \tag{36}$$

The first one can be reformulated into a different form

$$P'J(\alpha+\beta\lambda_i)Q' \;=\; E\;, \tag{37}$$

and from eqs.(35) and (37) we can further get

$$\begin{aligned}
\frac{\beta}{\alpha}P'JQ' &= \frac{1}{\alpha}E - P'Q' \\
&= \frac{1}{\alpha}E - Q'^{-1}J^{-1}(\alpha+\beta\lambda_i)Q' \\
&= Q'^{-1}M^{-1}(\frac{1}{\alpha}E - J(\frac{1}{\alpha+\beta\lambda_i}))MQ' \\
&= Q'^{-1}M^{-1}J(\frac{\beta\lambda_i}{\alpha(\alpha+\beta\lambda_i)})MQ'\;.
\end{aligned} \tag{38}$$

Here, M is an invertible matrix, and the theorem (6.2.25) in [20] is employed. Thus,

$$\begin{aligned}
\gamma P'JQ' &= Q'^{-1}M^{-1}J(\frac{\gamma\lambda_i}{\alpha+\beta\lambda_i})MQ' \\
&= SJ(\frac{\gamma\lambda_i}{\alpha+\beta\lambda_i})S^{-1}
\end{aligned} \tag{39}$$

with $S = Q'^{-1}M^{-1}$. Therefore, from (34)-(39) we get

$$O_2\left(\begin{array}{c} E \\ J \end{array}\right) = \left(\begin{array}{c} E \\ S\oplus_i J_{n_i}(\frac{\gamma\lambda_i}{\alpha+\beta\lambda_i})S^{-1} \end{array}\right)\;, \tag{40}$$

and hence

$$\left(\begin{array}{c} E \\ J' \end{array}\right) = \left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array}\right)\left(\begin{array}{c} E \\ J \end{array}\right)\;. \tag{41}$$

Q.E.D.

3.2 Classification on set $C_{n,l}$ with $n = N-1$

Having classified the $C_{N,l}$, now we proceed to the $C_{N-1,l}$ case. For every $(\Gamma_1,\Gamma_2)\in C_{N-1,\,l}$, we can find an ILO T which implements the following transformation

$$\left(\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array}\right)\left(\begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array}\right) = \left(\begin{array}{c} \Gamma_1' \\ \Gamma_2' \end{array}\right)\;, \tag{42}$$

where $r(\Gamma_1') = N-1$ and $r(\Gamma_2') = l$. Then in principle one can find ILOs P_1 and Q_1, which transform the matrix vector (Γ_1',Γ_2') into the form (Λ,Γ_2'') where Λ is a $N\times N$ diagonal matrix with N-1 nonzero elements of 1 and one zero, and Γ_2'' is another $N\times N$ matrix in

a specific form. To be more explicit, taking $N=6$ as an example (but the procedure is N independent) the above mentioned procedure tells

$$\Lambda = P\Gamma_1' Q \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \tag{43}$$

$$\Gamma_2'' = P\Gamma_2' Q \quad = \quad \left(\begin{array}{ccc|ccc} \times & \times & \times & 0 & \times & 0 \\ \times & \times & \times & 0 & \times & 0 \\ \times & \times & \times & 0 & \times & 0 \\ \hline \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} A & c \\ r & B_3 \end{pmatrix} , \tag{44}$$

where A, B_3, c and r are submatrices of Γ_2'' partitioned by vertical and horizontal lines; $\times$ means no constraint on the corresponding element. Note that for the case of N-by-N matrix, Γ_2'' can also be partitioned into a similar form as (44), with the lower right block to be still the B_3 (see Appendix A for details).

In general, there are four different choices for c and r, i.e., 1) $c=0,\ r=0$; 2) $c\neq 0,\ r=0$; 3) $c=0,\ r\neq 0$; 4) $c\neq 0,\ r\neq 0$. In these cases Γ_2'' can be further simplified through a series of elementary operations, e.g. denoted P_k and Q_k, which enables

$$\begin{pmatrix} \Lambda \\ \Gamma_2''^{cr} \end{pmatrix} = \begin{pmatrix} P_k \Lambda Q_k \\ P_k \Gamma_2'' Q_k \end{pmatrix} . \tag{45}$$

Here, the superscripts c and r stand for different choices mentioned in above, and the $\Gamma_2''^{cr}$ read

$$\Gamma_2''^{00} \quad = \quad \begin{pmatrix} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} , \quad \Gamma_2''^{10} = \left(\begin{array}{ccc|ccc} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) ,$$

$$\Gamma_2''^{01} \quad = \quad \left(\begin{array}{ccc|ccc} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) , \quad \Gamma_2''^{11} = \left(\begin{array}{ccc|ccc} \times & 0 & \times & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) . \tag{46}$$

In the case of $\Gamma_2''^{00}$, it has already been in the form of $\begin{pmatrix} A & 0 \\ 0 & B_3 \end{pmatrix}$. In the other three cases we can repartition the submatrices, like

$$\Gamma_2''^{10} = \left(\begin{array}{cc|cccc} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} A & c \\ 0 & B_4 \end{pmatrix}, \tag{47}$$

$$\Gamma_2''^{01} = \left(\begin{array}{cc|cccc} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 & 0 \\ \hline \times & \times & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} A & 0 \\ r & B_4 \end{pmatrix}, \tag{48}$$

$$\Gamma_2''^{11} = \left(\begin{array}{c|ccccc} \times & 0 & \times & 0 & 0 & 0 \\ \hline \times & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} A & c \\ r & B_5 \end{pmatrix}, \tag{49}$$

where n in B_n means the dimension of the matrix. This procedure results in the enlargement of the blocks B_n, the shrink of blocks A, and the emergence of new types of off diagonal blocks c and r.

By the same procedure, one can further simplify, enlarge B_n and shrink A, the $\Gamma_2''^{cr}$ of the forms (47-49). Finally the Γ_2'' in (44) may arrive at the form of

$$\Gamma_2'' \sim \begin{pmatrix} A & 0 \\ 0 & B_n \end{pmatrix} = \begin{pmatrix} DJD^{-1} & 0 \\ 0 & B_n \end{pmatrix} \tag{50}$$

with different kinds of B_ns, correspondingly. Here, J is the Jordan canonical form of A, and D is an invertible matrix. Every Γ_2'' matrix has its own specific form of B_n block. Note that B_n may be recursively enlarged to be the whole matrix of Γ_2'', i.e., $n = N$. In all, for every $(\Gamma_1, \Gamma_2) \in C_{N-1,l}$, there exists an ILO transformation, like

$$\begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix} = T \otimes P \otimes Q \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}. \tag{51}$$

Here, the $\Gamma = \begin{pmatrix} J & 0 \\ 0 & B_n \end{pmatrix}$, $P = \prod_i P_i$ and $Q = \prod_i Q_i$, where P_i, Q_i stand for those operators in Eqs. (44), (45), and (50). From Eq.(51) the subset $c_{N-1,\, l}$, defined as

$$c_{N-1,\, l} = \{(\Lambda, \Gamma)|\, r(\Gamma) = l; \Gamma = \begin{pmatrix} J & 0 \\ 0 & B_n \end{pmatrix};\ (\Lambda, \Gamma) \in C_{N-1,\, l}\}\ , \tag{52}$$

is equivalent to $C_{N-1,\, l}$ under the joint invertible transformations of T, P, and Q, which means that the classification on $C_{N-1,\, l}$ can be simply performed on $c_{N-1,\, l}$.

Similar as (27) we find

$$\mathrm{Det}(\rho_j) = \prod_i [\sum_{m=0}^{n_i} \frac{(1+|\lambda_i|^2)^m}{(n_i - m)!}\ f_{m+1}^{(n_i-m)}(x)\Big|_{x=0}\ ] \cdot 2^{\,r(B_n)-1} \neq 0\ , \tag{53}$$

where $j = \psi_1, \psi_2$, and n_i, λ_i is defined as the Jordan blocks in Eq.(50), and conclude that all the states in $c_{N-1,l}$ are truly entangled in the form of $2 \times N \times N$.

Theorem 2 $\forall\ (\Lambda, \Gamma) \in c_{N-1,\, l}$, *the set* $c_{N-1,\, l}$ *is of the classification of* $C_{N-1,\, l}$ *under SLOCC:*
(i) *suppose two states in* $C_{N-1,\, l}$ *are SLOCC equivalent, they can then be transformed into the same matrix vector* (Λ, Γ)*;*
(ii) *the matrix vector in* $c_{N-1,l}$ *is unique up to a nonzero classification irrelevant parameter, i.e., provided* (Λ, Γ') *is SLOCC equivalent with* (Λ, Γ)*, then* $(\Lambda, \Gamma') = (\Lambda, \kappa\Gamma)$ *in the sense of* J' *equalling to* J *as in theorem 1 while* B_n' *being exactly the same as* B_n*.*

Proof:
(i) According to the property of transitivity in SLOCC transformation, this proposition should be true.

(ii) Suppose

$$\begin{pmatrix} \Lambda \\ \Gamma' \end{pmatrix} = T' \otimes P' \otimes Q' \begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix}\ , \tag{54}$$

we first demonstrate that the three ILOs transformation T', P', Q' can always be fulfilled through two ILOs transformations P'', Q'', that is

$$\begin{aligned} T' \otimes P' \otimes Q' \begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix} &= P'' \otimes Q'' \begin{pmatrix} \Lambda \\ \kappa\Gamma \end{pmatrix} \\ &= \begin{pmatrix} P''\Lambda Q'' \\ P''\kappa\Gamma Q'' \end{pmatrix}\ . \end{aligned} \tag{55}$$

According to the definition of $c_{N-1,l}$ we can write (Λ, Γ) in the form of direct sums

$$\left(\begin{array}{c} \Lambda \\ \Gamma \end{array} \right) = \left(\begin{array}{c} \left(\begin{array}{cc} E & 0 \\ 0 & \Lambda' \end{array} \right) \\ \left(\begin{array}{cc} J & 0 \\ 0 & B_n \end{array} \right) \end{array} \right) , \tag{56}$$

where

$$\Lambda' = \left(\begin{array}{ccccc} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{array} \right) \tag{57}$$

is a square matrix, and its dimension equals to that of B_n. The transformation T'

$$\left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) \left(\begin{array}{c} \Lambda \\ \Gamma \end{array} \right) \tag{58}$$

can be decomposed into the following form

$$\left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) \left(\begin{array}{c} E \\ J \end{array} \right) , \tag{59}$$

$$\left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) \left(\begin{array}{c} \Lambda' \\ B_n \end{array} \right) , \tag{60}$$

according to the nature of direct sum.

For Eq.(59), from the proof of *theorem 1* one can find the following P_J, Q_J

$$\left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) \left(\begin{array}{c} P_J E Q_J \\ P_J J Q_J \end{array} \right) = \left(\begin{array}{c} E \\ J \end{array} \right) , \tag{61}$$

should exist. Here $J + \lambda E$ is taken to be equivalent to the J while $r(J + \lambda E) = r(J) = l$.

For Eq.(60), we take into account the two decomposed operations of T' separately. We have

$$\left(\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array} \right) \left(\begin{array}{c} \Lambda' \\ B_n \end{array} \right) = \left(\begin{array}{c} \alpha(\Lambda' + \sigma B_n) \\ \gamma B_n \end{array} \right) = \left(\begin{array}{cc} \alpha & 0 \\ 0 & \gamma \end{array} \right) \left(\begin{array}{c} \Lambda' + \sigma B_n \\ B_n \end{array} \right) \tag{62}$$

with $\sigma = \frac{\beta}{\alpha}$. As shown in Appendix B, there exist ILOs P_{Bn} and Q_{Bn} which satisfy

$$P_{Bn} \left(\begin{array}{c} \Lambda' + \lambda' B_n \\ B_n \end{array} \right) Q_{Bn} = \left(\begin{array}{c} \Lambda' \\ B_n \end{array} \right) . \tag{63}$$

And, further more we have

$$\left[\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array}\right] \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda' \\ B_n + \frac{\alpha\lambda}{\gamma}\Lambda' \end{pmatrix} . \tag{64}$$

There also exist ILOs P'_{Bn} and Q'_{Bn} which satisfy (see Appendix B)

$$P'_{Bn} \begin{pmatrix} \Lambda' \\ B_n + \lambda\Lambda' \end{pmatrix} Q'_{Bn} = \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix} . \tag{65}$$

Thus Eq.(60) becomes

$$\left[\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array}\right] \left[\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array}\right] \begin{pmatrix} P_B\Lambda' Q_B \\ P_B B_n Q_B \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \Lambda' \\ B_n \end{pmatrix} , \tag{66}$$

where $P_B = P'_{Bn}P_{Bn}$ and $Q_B = Q_{Bn}Q'_{Bn}$. By taking $P_0 = P_J \oplus \frac{1}{\alpha}P_B$ and $Q_0 = Q_J \oplus Q_B$, we have

$$\left[\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array}\right] \left[\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array}\right] \begin{pmatrix} P_0\Lambda Q_0 \\ P_0\Gamma Q_0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \Lambda' \end{pmatrix} \\ \begin{pmatrix} J & 0 \\ 0 & \frac{\gamma}{\alpha}B_n \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \Lambda' \end{pmatrix} \\ \kappa \begin{pmatrix} \frac{1}{\kappa}J & 0 \\ 0 & B_n \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \Lambda \\ \kappa\Gamma \end{pmatrix} . \tag{67}$$

Then, Eq.(54) can be expressed as

$$\begin{aligned} \begin{pmatrix} \Lambda \\ \Gamma' \end{pmatrix} &= P' \left[\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array}\right] \left[\begin{array}{cc} \alpha & \beta \\ 0 & \gamma \end{array}\right] \begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix} Q' \\ &= P'P_0^{-1} \begin{pmatrix} \Lambda \\ \kappa\Gamma \end{pmatrix} Q_0^{-1}Q' \\ &= \begin{pmatrix} P''\Lambda Q'' \\ P''\kappa\Gamma Q'' \end{pmatrix} , \end{aligned} \tag{68}$$

which is just Eq.(55).

Eq.(68) corresponds to two equations

$$\begin{cases} P''\Lambda Q'' = \Lambda \\ P''\kappa\Gamma Q'' = \Gamma' \end{cases} . \tag{69}$$

The equation $\Lambda = P''\Lambda Q''$ requires P'' and Q'' taking the following form

$$P'' = \begin{pmatrix} S & Y \\ 0 & p \end{pmatrix} ; \; Q'' = \begin{pmatrix} S^{-1} & 0 \\ X & q \end{pmatrix} , \tag{70}$$

respectively. Here, p and q are arbitrary complex numbers. Note that in order to guarantee P'' and Q'' to be non-singular matrices, p and q can not be zero.

The canonical form of Γ' in $c_{N-1,l}$ gives further constraints on the patterns of P'' and Q''. Of the Γ' and Γ, in which the B_3 takes the form of (44), they can be generally expressed as

$$\begin{pmatrix} \times & \times & \times & 0 & v_1 & 0 \\ \times & \times & \times & 0 & v_2 & 0 \\ \times & \times & \times & 0 & v_3 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} . \tag{71}$$

In fact any elements in set $c_{N-1,l}$ takes the same patten (71). From (70), P'' and Q'' take the forms of

$$P'' = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & y_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & y_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & y_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & y_4 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & y_5 \\ 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}, \quad Q'' = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & q \end{pmatrix}, \tag{72}$$

where $S = \{a_{ij}\}_{5\times 5} = \{b_{ij}\}^{-1}_{5\times 5}$. From (71) and constraint $\Gamma' = P''\kappa\Gamma Q''$, P'' and Q'' read

$$P'' = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & y_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & y_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & y_3 \\ 0 & 0 & 0 & p\kappa & 0 & y_4 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1/q\kappa & y_5 \\ 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}, \quad Q'' = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & 0 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & 0 \\ 0 & 0 & 0 & 1/p\kappa & 0 & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & q\kappa & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & q \end{pmatrix}. \tag{73}$$

We notice that if P'' and Q'' are invertible, then the upper-left sub-matrices $\{a_{ij}\}_{3\times 3}$ and $\{b_{ij}\}_{3\times 3}$ should also be invertible, due to the fact that any invertible matrix in the form $\begin{pmatrix} X & 0 \\ W & Y \end{pmatrix}$ has an inverse $\begin{pmatrix} X^{-1} & 0 \\ Z & Y^{-1} \end{pmatrix}$ with arbitrary matrices X and Y being also invertible. From (69) one may infer that in $P''\kappa\Gamma Q''$ only the block $\{a_{ij}\}_{3\times 3}$ acts on vector $v = \{v_i, i = 1, 2, 3\}$ in Γ of (71). Since no invertible operator can transform a nonzero vector into a null one, therefore $v = 0$ and $v \neq 0$ determine two ILO inequivalent cases for Γ. Thus we see that if Γ' and Γ in $c_{N-1,l}$ satisfy Eq.(69), the identity of their B_3 sub-matrices leads to the identity of their B_4 sub-matrices. Or in other words, for two matrices Γ and Γ', if their sub-matrices B_3 are the same, while their B_4 sub-matrices are different, then they should be ILO inequivalent, like (47) and (48).

The above analysis for B_3 is applicable to other sub-matrices B_n with $n > 3$, e.g., the forms of B_4 and B_5 in Γ_2'' in Eqs.(47)-(49). Generally every nonzero element in B_n will transform one column or one row of P'' and Q'' into a unit vector. In the end, under the constraint (69), if two matrices Γ and Γ' have the same B_i, then they must possess the same B_{i+1}. According to theorem 1 we then have

$$P''\kappa \begin{pmatrix} J(\lambda_i) & 0 \\ 0 & B_n \end{pmatrix} Q'' = \begin{pmatrix} SJ(\kappa\lambda_i)S^{-1} & 0 \\ 0 & B_n \end{pmatrix} , \tag{74}$$

that is

$$\begin{pmatrix} \Lambda \\ \Gamma' \end{pmatrix} = \begin{pmatrix} \Lambda \\ \Gamma \end{pmatrix} . \tag{75}$$

(Notice that there exists a special case in which B matrix takes up the whole Γ and then the (59) does not exist. The elements in the pivot of T' now can have zeros. In this situation, the only modification of the above proofing process is by adding two more ILOs P_t and Q_t which can reverse the flip of the (Λ', B) induced by the zero elements in the pivot of T', see Appendix C for details)
Q.E.D.

3.3 Classification on set $C_{n,l}$ with $n = N - i$

The same procedure can be directly applied to the $C_{N-2,\,l}$ case, and so on. Taken here again the $N = 6$ case as an example, following the construction procedure in Eq.(44) it is easy to obtain the canonical form of (Λ, Γ) in $c_{6-2,\,l}$

$$\Lambda = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) , \tag{76}$$

$$\Gamma = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) . \tag{77}$$

In this case, it is obvious that l can not be smaller than 4. Rescale the matrices (76) and (77) according to the partition lines we have

$$\Lambda = \begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{78}$$

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E \\ E & 0 & 0 \end{pmatrix} . \tag{79}$$

Here Γ is just like B_3 in Eq.(44). Thus the classification of set $c_{N-i,l}$ with $i > 1$ can be reduced to the classification of $c_{N-1,l}$ in principle. From (44) and (77) we notice that a proposition(inequality) of n, l in $c_{n,l}$ should exist, i.e.

$$2(N-n) \leq l \leq n , \tag{80}$$

when $n < N$. Eq.(80) is a constraint on the rank of matrix-pair which represents the true entangled state of $2 \times N \times N$.

Now we have fully classified all the truly entangled classes of $2 \times N \times N$ state. A truly entangled state of $2 \times N \times N$ must line in one of the sets $C_{N-i,l}$ (or $C_{n,l}$). According to (80) we can obtain a restriction on the values of n,

$$\frac{2N}{3} \leq \quad n \quad < N \tag{81}$$

when $i > 0$, and from the arguments above (26) we know

$$n = N \ , \ 0 < l < N \tag{82}$$

when $i = 0$. From those two theorems proved in this section we know that the mapping of $C_{N-i,l} \mapsto c_{N-i,l}$ determines all the true entanglement classes in $C_{N-i,l}$.

4 Examples

According to above explanation, the classification of the entangled state $2 \times N \times N$ may be accomplished by repeatedly taking the above introduced procedures. To be more specific and for readers convenience, in the following we completely classify the $2 \times 2 \times 2$ and $2 \times 4 \times 4$ pure states by using this novel method as examples.

For $N = 2$, i.e., three qubits states, there is only one inequivalent set $c_{N=2,l=1}$ (the case $c_{1,1}$ does not exist in the three qubits true entanglement state due to proposition

(80)). There are two inequivalent Jordan forms for 2×2 matrices with rank one, and thus two inequivalent classes in $c_{2,1}$ which correspond to the GHZ and W states separately [7]

$$\mathrm{GHZ} \;:\quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\; J = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}, \tag{83}$$

$$\mathrm{W} \;:\quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\; J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{84}$$

For $N = 4$, from (80)-(82) the inequivalent sets are

$$c_{N,l} = c_{4,1}\,, c_{4,2}\,, c_{4,3} \tag{85}$$

$$c_{N-1,l} = c_{3,2}\,, c_{3,3}\;. \tag{86}$$

Due to (81), there is no $c_{N-i,l}$ sets in truly entangled states for $N = 4$ when $i \geq 2$. In the case of $c_{4,l}$ all inequivalent classes have the form of $\begin{pmatrix} E \\ J \end{pmatrix}$, where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{87}$$

Hence, we can distinguish them just by virtue of J's pattern. There are two classes in set $c_{4,1}$, i.e.,

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \tag{88}$$

six classes in set $c_{4,2}$, the

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \tag{89}$$

and five classes in set $c_{4,3}$, the

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{90}$$

In the case of $c_{3,l}$, every class has the form of $\left(\begin{array}{c}\Lambda \\ \Gamma\end{array}\right)$ where

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{91}$$

From theorem 2 in Section 3.2 we can simply classify the set $c_{3,2}$ by the pattern of Γ matrix. And, from the measure in constructing matrices B_3 and B_4 in the same section, we find that there is one class in $c_{3,2}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} , \tag{92}$$

and two classes in $c_{3,3}$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \tag{93}$$

Altogether, there are 16 genuine entanglement classes in $2 \times 4 \times 4$ states, which agrees with what obtained in Ref.[12]. From (93) one can easily conclude that the permutation of the two 4 dimension partites are sorted into different classes, which was noticed in [12].

In above examples we enumerate various distinct classes of the $2 \times 2 \times 2$ and $2 \times 4 \times 4$ states, each with a representative state. In fact there are still reducible parameters in the representative states, the eigenvalues of the Jordan form, e.g. the λs in (89). These parameters can be sorted into two categories: one with only redundant parameters, which can be eliminated out of the state through ILOs; another possesses non-local parameters which can not be eliminated through the ILOs and will keep on staying in the entangled state as free parameters. For the first case, we take one typical class in set $c_{5,2}$ as an example. The first three diagrams of Fig.(2) exhibits the procedure of how the redundant parameters being eliminated through elementary operations. Multiplying the vertical or horizontal A-B plane of the cubic form in (II) by a factor of $\frac{1}{\lambda_2}$ (elementary operation **type 2**), we can get the form of diagram (III). The multiplication of the back plane of the cubic by factor of $\frac{\lambda_2}{1-\lambda_2}$ (elementary operation **type 2**) will transform the parameter $\frac{1-\lambda_2}{\lambda_2}$ from node B to nodes C, D, E, which is represented by the arrow between B and C. Then, the parameters can be easily eliminated by elementary operations in three vertical planes containing nodes C, D, and E, respectively.

For the second case, different from the situations shown in first three diagrams, the flow of parameters in performing the elementary operations may form loops, like shown

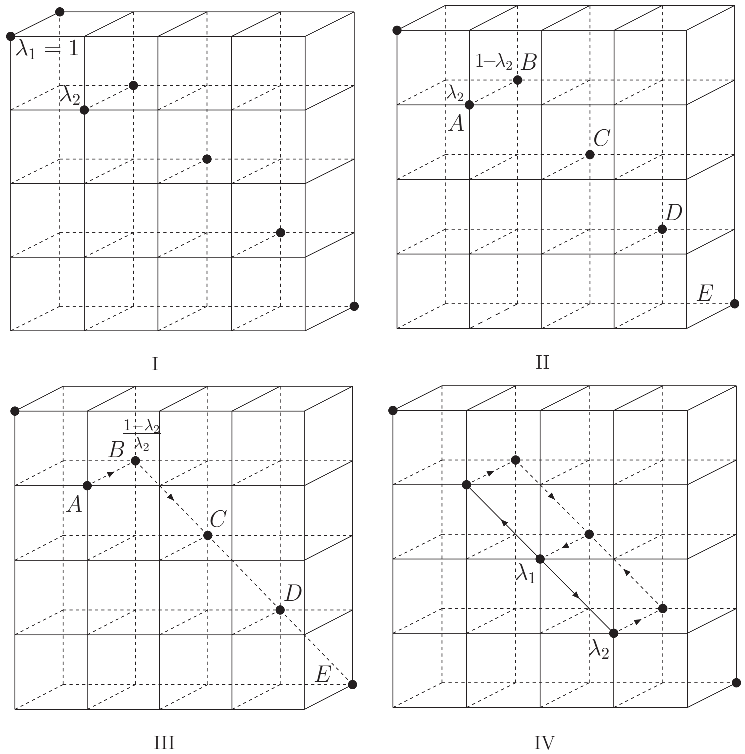

Figure 2: The pictorial procedure of eliminating the parameters in entanglement state. The plain nodes represent 0 and the solid dark nodes represent 1 if not further specified. (I) represents the initial state in cubic form; (II) is a transformed form from (I) by subtracting the front plane from the back one; (III) shows the elimination procedure of the parameter on node A. (IV) shows the case with two free parameters.

in diagram IV of Fig.(2) for one typical class in set $c_{5,4}$ as an example. As long as the loop(s) appears, the non-local parameter(s) in the entanglement state remains, and vice versa. The number of non-local parameters therefore equals to the number of the loops. It is worth to mention that although these parameters are free ones, they satisfy certain relations in giving out the equivalent classes, like

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \sim \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \sim \begin{pmatrix} 1-\lambda_1 \\ 1-\lambda_2 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \end{pmatrix} \sim \begin{pmatrix} \frac{1}{\lambda_1} \\ \frac{\lambda_2}{\lambda_1} \end{pmatrix} \tag{94}$$

for $c_{5,4}$. The situation may become complicated as the number of parameters increase. To get a deeper insight of the behavior of those non-local parameters, there still needs a lot of work.

5 Summary and Conclusions

In conclusion, we put forward a novel method in classifying the entangled pure states of $2 \times N \times N$. A remarkable feature of our method in different from what existed in the literature is that it does not need to classify the lower dimension cases first [11, 12]. We find in practice that this method in classifying the $2\times N\times N$ tri-partite entanglement state is quite straightforward. Since the software for Jordan decomposition is available, this new method can be applied to the classification of a given state via automatic computer calculation, which is very important as the partite dimension N tends to be large. Last, but not least, in this work a pictorial configuration of the entanglement states on the grids is proposed , which gives an intuitive demonstration for the non-local parameters, and is efficient in eliminating redundant parameters.

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Appendix

A The Construction of B matrix

$$P_1\Gamma_1'Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{95}$$

$$P_1\Gamma_2'Q_1 = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} & \gamma_{35} & \gamma_{36} \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} & \gamma_{45} & \gamma_{46} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & \gamma_{55} & \gamma_{56} \\ \gamma_{61} & \gamma_{62} & \gamma_{63} & \gamma_{64} & \gamma_{65} & \gamma_{66} \end{pmatrix}. \tag{96}$$

A direct observation on Eq.(96) tells that γ_{66} must be zero, otherwise one can find invertible operators P_x and Q_x which enable

$$P_xP_1\Gamma_1'Q_1Q_x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{97}$$

$$P_xP_1\Gamma_2'Q_1Q_x = \begin{pmatrix} \gamma_{11x} & \gamma_{12x} & \gamma_{13x} & \gamma_{14x} & \gamma_{15x} & 0 \\ \gamma_{21x} & \gamma_{22x} & \gamma_{23x} & \gamma_{24x} & \gamma_{25x} & 0 \\ \gamma_{31x} & \gamma_{32x} & \gamma_{33x} & \gamma_{34x} & \gamma_{35x} & 0 \\ \gamma_{41x} & \gamma_{42x} & \gamma_{43x} & \gamma_{44x} & \gamma_{45x} & 0 \\ \gamma_{51x} & \gamma_{52x} & \gamma_{53x} & \gamma_{54x} & \gamma_{55x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{98}$$

Given λ_i are the eigenvalues of submatrix $\{\gamma_{ijx}\}_{5\times 5}$ and $\lambda \neq -\lambda_i$, we will find that $r(P_xP_1\Gamma_2'Q_1Q_x + \lambda P_xP_1\Gamma_1'Q_1Q_x) = N > N-1$. This contradicts to requirement that the maximum rank of $(t_{11}\Gamma_1 + t_{12}\Gamma_2)$ is $N-1$.

Let $\gamma_{66}=0$, Eqs.(95) and (96) become

$$P_1\Gamma_1'Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{99}$$

$$P_1\Gamma_2'Q_1 = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & \gamma_{25} & \gamma_{26} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \gamma_{34} & \gamma_{35} & \gamma_{36} \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} & \gamma_{45} & \gamma_{46} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} & \gamma_{54} & \gamma_{55} & \gamma_{56} \\ \gamma_{61} & \gamma_{62} & \gamma_{63} & \gamma_{64} & \gamma_{65} & 0 \end{pmatrix}. \tag{100}$$

Since we are considering the true entanglement of $2\times N\times N$ states, neither the last column nor the last row of the matrix in Eq.(100) can be completely zero. There exist ILOs P_2, Q_2 which satisfy the following equations

$$P_2P_1\Gamma_1'Q_1Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{101}$$

$$P_2P_1\Gamma_2'Q_1Q_2 = \begin{pmatrix} \gamma_{11}' & \gamma_{12}' & \gamma_{13}' & \gamma_{14}' & \gamma_{15}' & 0 \\ \gamma_{21}' & \gamma_{22}' & \gamma_{23}' & \gamma_{24}' & \gamma_{25}' & 0 \\ \gamma_{31}' & \gamma_{32}' & \gamma_{33}' & \gamma_{34}' & \gamma_{35}' & 0 \\ \gamma_{41}' & \gamma_{42}' & \gamma_{43}' & \gamma_{44}' & \gamma_{45}' & 0 \\ \gamma_{51}' & \gamma_{52}' & \gamma_{53}' & \gamma_{54}' & \gamma_{55}' & 1 \\ \gamma_{61}' & \gamma_{62}' & \gamma_{63}' & \gamma_{64}' & \gamma_{65}' & 0 \end{pmatrix}. \tag{102}$$

An invertible operator Q_3

$$Q_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\gamma_{51}' & -\gamma_{52}' & -\gamma_{53}' & -\gamma_{54}' & -\gamma_{55}' & 1 \end{pmatrix} \tag{103}$$

makes

$$
P_2 P_1 \Gamma_1' Q_1 Q_2 Q_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{104}
$$

$$
P_2 P_1 \Gamma_2' Q_1 Q_2 Q_3 = \begin{pmatrix} \gamma_{11}' & \gamma_{12}' & \gamma_{13}' & \gamma_{14}' & \gamma_{15}' & 0 \\ \gamma_{21}' & \gamma_{22}' & \gamma_{23}' & \gamma_{24}' & \gamma_{25}' & 0 \\ \gamma_{31}' & \gamma_{32}' & \gamma_{33}' & \gamma_{34}' & \gamma_{35}' & 0 \\ \gamma_{41}' & \gamma_{42}' & \gamma_{43}' & \gamma_{44}' & \gamma_{45}' & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \gamma_{61}' & \gamma_{62}' & \gamma_{63}' & \gamma_{64}' & \gamma_{65}' & 0 \end{pmatrix}. \tag{105}
$$

Here, γ_{65}' must be zero also, otherwise to keep the form of (104) unchanged the matrix in Eq.(105) can be transformed into

$$
\begin{pmatrix} \gamma_{11}'' & \gamma_{12}'' & \gamma_{13}'' & \gamma_{14}'' & 0 & 0 \\ \gamma_{21}'' & \gamma_{22}'' & \gamma_{23}'' & \gamma_{24}'' & 0 & 0 \\ \gamma_{31}'' & \gamma_{32}'' & \gamma_{33}'' & \gamma_{34}'' & 0 & 0 \\ \gamma_{41}'' & \gamma_{42}'' & \gamma_{43}'' & \gamma_{44}'' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{106}
$$

Clearly this will lead to the same contradiction as γ_{66} does in (96) and (98). Thus Eq.(105) becomes

$$
P_2 P_1 \Gamma_2' Q_1 Q_2 Q_3 = \begin{pmatrix} \gamma_{11}' & \gamma_{12}' & \gamma_{13}' & \gamma_{14}' & \gamma_{15}' & 0 \\ \gamma_{21}' & \gamma_{22}' & \gamma_{23}' & \gamma_{24}' & \gamma_{25}' & 0 \\ \gamma_{31}' & \gamma_{32}' & \gamma_{33}' & \gamma_{34}' & \gamma_{35}' & 0 \\ \gamma_{41}' & \gamma_{42}' & \gamma_{43}' & \gamma_{44}' & \gamma_{45}' & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \gamma_{61}' & \gamma_{62}' & \gamma_{63}' & \gamma_{64}' & 0 & 0 \end{pmatrix}. \tag{107}
$$

Applying the same procedure to the last row as we have performed to the last column,

we have

$$\Lambda = P\Gamma_1' Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{108}$$

$$\Gamma_2'' = P\Gamma_2' Q = \left(\begin{array}{ccc|ccc} \times & \times & \times & 0 & c_{15} & 0 \\ \times & \times & \times & 0 & c_{25} & 0 \\ \times & \times & \times & 0 & c_{35} & 0 \\ \hline r_{31} & r_{32} & r_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) = \begin{pmatrix} A & c \\ r & B_3 \end{pmatrix}, \tag{109}$$

where $P = \prod_i P_i$ and $Q = \prod_i Q_i$ are sequences of invertible operators P_i and Q_i, respectively.

B The Superpositions of Λ' and B_n

Eq.(63) can be written in the following matrix equations

$$\begin{cases} P_{B_n}(\Lambda_n' + \lambda B_n)Q_{B_n} = \Lambda_n' \\ P_{B_n} B_n Q_{B_n} = B_n \end{cases}. \tag{110}$$

Here, for the sake of clarity, we label the Λ' with subscript n to indicate its dimension.

Following, we show inductively that the invertible matrices P_{B_n} and Q_{B_n} can always be constructed.

First, in case $n = 1$, then $\Lambda' = 0$, $B_1 = 0$, the construction of invertible operators P_{B_1}, Q_{B_1} Eq.(110) is trivial.

In the case of $n = 2$, Eq.(110) becomes

$$\begin{cases} P_{B_2}\begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix} Q_{B_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ P_{B_2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Q_{B_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{cases}. \tag{111}$$

$P_{B_2} = E$ and Q_{B_2} of the form $\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}$ satisfy the above equations.

Suppose for arbitrary n (110) is true, we show that $P_{B_{n+1}}, Q_{B_{n+1}}$ can also be constructed, satisfying

$$\left\{ \begin{array}{ccc} P_{B_{n+1}}(\Lambda'_{n+1} + \lambda B_{n+1})Q_{B_{n+1}} & = & \Lambda'_{n+1} \\ P_{B_{n+1}} B_{n+1} Q_{B_{n+1}} & = & B_{n+1} \end{array} \right. . \tag{112}$$

Here, either

$$B_{n+1} = \begin{pmatrix} 0 & r \\ 0 & B_n \end{pmatrix} \tag{113}$$

or

$$B_{n+1} = \begin{pmatrix} 0 & 0 \\ c & B_n \end{pmatrix} , \tag{114}$$

where $r(B_n) = n-1$, the ranks of $\begin{pmatrix} r \\ B_n \end{pmatrix}$ and $\begin{pmatrix} c & B_n \end{pmatrix}$ are n. And,

$$\Lambda'_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda'_n \end{pmatrix} . \tag{115}$$

In one example of $n = 5$, Λ' and B can be expressed as follows

$$\Lambda'_{5+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{5+1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} , \tag{116}$$

where $B_{5+1} = \begin{pmatrix} 0 & r \\ 0 & B_5 \end{pmatrix}$, $r = (0, 1, 0, 0, 0)$.

The operator $P_{B_{n+1}}$ and $Q_{B_{n+1}}$ can be constructed as follows

$$P_{B_{n+1}} = \begin{pmatrix} 1 & X \\ 0 & E \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P_{B_n} \end{pmatrix} , \tag{117}$$

$$Q_{B_{n+1}} = \begin{pmatrix} 1 & 0 \\ 0 & Q_{B_n} \end{pmatrix} \begin{pmatrix} 1 & -Y \\ 0 & E \end{pmatrix} , \tag{118}$$

where $Y = \lambda r Q_{B_n} + X\Lambda'_n$. Because the rank of $\begin{pmatrix} 0 & r \\ 0 & B_n \end{pmatrix}$ is unchanged under the invertible transformation, we can always find such invertible operator $\begin{pmatrix} 1 & X \\ 0 & E \end{pmatrix}$ which satisfies

$$\begin{pmatrix} 1 & X \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & rQ_{B_n} \\ 0 & B_n \end{pmatrix} = \begin{pmatrix} 0 & r \\ 0 & B_n \end{pmatrix} . \tag{119}$$

It is then easy to verify Eq.(112) using Eqs.(117,118). Note that the $P_{B_{n+1}}$ and $Q_{B_{n+1}}$ constructed above correspond to the case of Eq.(113), for the case of (114) the procedure is similar.

Along the same line, it can also be found that there exist such invertible operators P'_{B_n}, Q'_{B_n} that

$$\left\{ \begin{array}{ccc} P'_{B_n}\Lambda'_n Q'_{B_n} & = & \Lambda'_n \\ \\ P'_{B_n}(B_n + \lambda\Lambda'_n)Q'_{B_n} & = & B_n \end{array} \right. . \tag{120}$$

C The flip of Λ'_n and B_n

Using the Eq.(110) and Eq.(120), we show that there exist the following invertible matrices P_t, Q_t which flip the Λ'_n and B_n, like

$$\left(\begin{array}{c} P_t\Lambda'_n Q_t \\ P_t B_n Q_t \end{array} \right) = \left(\begin{array}{c} B_n \\ \Lambda' \end{array} \right) . \tag{121}$$

Provided

$$\left\{ \begin{array}{ccc} P_{B_n}(\lambda)(\Lambda'_n + \lambda B_n)Q_{B_n}(\lambda) & = & \Lambda'_n \\ \\ P_{B_n}(\lambda)B_n Q_{B_n}(\lambda) & = & B_n \end{array} \right. , \tag{122}$$

and

$$\left\{ \begin{array}{ccc} P'_{B_n}(\lambda)\Lambda'_n Q'_{B_n}(\lambda) & = & \Lambda'_n \\ \\ P'_{B_n}(\lambda)(B_n + \lambda\Lambda'_n)Q'_{B_n}(\lambda) & = & B_n \end{array} \right. , \tag{123}$$

it can be found that

$$P_t = P_{B_n}(\lambda)P'_{B_n}(-\frac{1}{\lambda})P_{B_n}(\lambda) \ , \quad Q_t = Q_{B_n}(\lambda)Q'_{B_n}(-\frac{1}{\lambda})Q_{B_n}(\lambda) \tag{124}$$

enables

$$\left(\begin{array}{c} P_t\Lambda'_n Q_t \\ P_t B_n Q_t \end{array} \right) = \left(\begin{array}{c} -\lambda B_n \\ \frac{1}{\lambda}\Lambda'_n \end{array} \right) , \tag{125}$$

which is equivalent to (121) up to irrelevant coefficients.

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